Equimomental Systems and Robot Dynamics

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Look at three old problems from a more modern viewpoint.

- How many rigidly connected point masses are needed so that the system has the same inertia properties as an arbitrary rigid body?
- Design a serial robot arm with constant mass matrix.
- Rotor balancing.

Linked by geometry of a Veronese variety, 2-uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$. 

The Inertia Matrix

Two rigid bodies or systems of point masses are said to be **equimomentally** if they have the same inertia matrices (or one can be transformed to the other by a rigid change of coordinates). The $6 \times 6$ inertia matrix has the form,

$$N = m \begin{pmatrix} \mathbb{I} & C \\ C^T & I_3 \end{pmatrix}$$

where,

- $m$ is the total mass of the body and $I_3$ is the $3 \times 3$ identity matrix
- $C$ is the position of the body’s centre of mass written as an anti-symmetric $3 \times 3$ matrix
- $\mathbb{I}$ is the usual $3 \times 3$ inertia matrix of the body
Here more convenient to use a different representation of the inertia,

\[
\tilde{\Xi} = m \begin{pmatrix}
\frac{1}{2}(-I_{xx} + I_{yy} + I_{zz}) & -l_{xy} & -l_{xz} & c_x \\
-l_{xy} & \frac{1}{2}(I_{xx} - I_{yy} + I_{zz}) & -l_{yz} & c_y \\
-l_{xz} & -l_{yz} & \frac{1}{2}(I_{xx} + I_{yy} - I_{zz}) & c_z \\
c_x & c_y & c_z & 1
\end{pmatrix}
\]

where \( m \) is the mass of the body as above, \( c_x \) and so forth, are the components of the centre of mass and \( l_{xy} \) etc. are the components of the \( 3 \times 3 \) inertia matrix.

Clearly, two bodies are equimomentential if and only if their \( 4 \times 4 \) inertia matrices are the same.
Point Masses

The $4 \times 4$ inertia matrix of a point with mass $m$, located at $\mathbf{p} = (p_x, p_y, p_z)^T$ will be,

$$\tilde{\Xi} = m \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} \begin{pmatrix} p_x & p_y & p_z & 1 \end{pmatrix}$$

Assume the point is in the projective space $\mathbb{P}^3$ with homogeneous coordinates, $\mathbf{p} = (p_x : p_y : p_z : p_0)$, then we can write,

$$\tilde{\Xi} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ p_0 \end{pmatrix} \begin{pmatrix} p_x & p_y & p_z & p_0 \end{pmatrix}$$

(mass not needed here).
The Veronese Variety

Consider the space of all $4 \times 4$ symmetric matrices, there is a 10-dimensional vector space of these matrices. Now if we ignore an overall scaling of the matrices the space of these matrices is a 9-dimensional projective space.

The $4 \times 4$ inertia matrices form an open set in this $\mathbb{P}^9$. Not all of $\mathbb{P}^9$ since inertia matrices are positive definite.

Point masses lie on the 3-D Veronese variety of rank 1 symmetric matrices.
Four Point Masses

Theorem, probably due to Routh $\sim$ 1870: *there is a system of four point masses of equal mass, equimoment to a general rigid body.*

In terms of the Veronese variety this implies that (almost) any point in $\mathbb{P}^9$ lies on a 3-plane which meets the Veronese variety in 4 points.

Proof—Suppose the mass of the body is $m$. Take 4 point masses each with mass $m/4$, and place them at the vertices of a regular tetrahedron. The extended position vectors of the points will be,

$$
\tilde{\mathbf{p}}_1 = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_2 = \begin{pmatrix} \frac{2\sqrt{2}}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_3 = \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_4 = \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}.
$$
Notice that these extended vectors satisfy the relations, \( \tilde{p}_i^T \tilde{p}_j = 0 \) when \( i \neq j \), and \( \tilde{p}_i^T \tilde{p}_i = 4 \) for \( i = 1, \ldots, 4 \). If these 4 points all have masses \( m/4 \) then the \( 4 \times 4 \) inertia matrix of the system will be,

\[
\tilde{\Xi} = \frac{m}{4} \sum_{i=1}^{4} \tilde{p}_i \tilde{p}_i^T = m \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Proof — concluded

Choose coordinates so that the $6 \times 6$ inertia matrix is diagonal, origin at centre of mass, axes aligned with principal axes of inertia. In this coordinate system $4 \times 4$ inertia matrix has the form,

$$
\tilde{\Xi} = m \begin{pmatrix}
a^2 & 0 & 0 & 0 \\
0 & b^2 & 0 & 0 \\
0 & 0 & c^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

where $a$, $b$ and $c$ related to the principal radii of gyration. The original inertia matrix can be duplicated by subjecting the 4 point-masses to a non-rigid transformation,

$$
\tilde{p}_i' = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \tilde{p}_i, \quad i = 1, 2, 3, 4
$$
Remarks

Notice that we could also have subjected the point-masses to an 4-D orthogonal transformation before the non-linear transformation. Suppose $U \in O(4)$ and

$$D = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$\frac{m}{4} \sum_{i=1}^{4} DU \tilde{p}_i \tilde{p}_i^T U^T D^T = mDI_4 U^T D^T = mDI_4 D^T = \Xi$$

This implies that the Veronese variety has a 6-parameter family of secant 3-plane through any point in $\mathbb{P}^9$. Since, $\dim(O(4)) = 6$. 
Can summarise equations of motion for a serial robot as,

\[ M_{ij} \ddot{\theta}_j + C_{ijk} \dot{\theta}_j \dot{\theta}_k = \tau_i \]

Here summation over repeated indices implied, for 6-joint robot range of sum is 1, \ldots, 6. \( \theta_i \) is the \( i \)th joint angle and \( \tau_i \) the torque applied by the motor at joint \( i \).

\( M_{ij} \) is the generalised mass matrix of the robot, when its elements are constant the terms \( C_{ijk} \) vanish.

(For simplicity no gravity here).
Elements of the Mass Matrix

The elements of the mass matrix are given by,

\[ M_{ij} = \begin{cases} 
  s_i^T (N_i + \cdots + N_6) s_j, & \text{if } i \geq j, \\
  s_j^T (N_j + \cdots + N_6) s_i, & \text{if } i < j,
\end{cases} \]

where \( N_j \) is the \( 6 \times 6 \) inertia matrix of the \( j \)th link and \( s_i \) are the twists corresponding to axis of the \( i \)th joint,

\[ s_i = \begin{pmatrix} \omega_i \\ v_i \end{pmatrix} \]

in partitioned form with \( \omega_i \) the angular velocity and \( v_i \) linear velocity.

Can show that the mass matrix will be constant if and only if the composite inertia of all the links above the \( i \)th joint are symmetrical about the \( i \)th joint.
Symmetry and Balancing

Here “symmetrical about an axis” means “is equimoment to a cylinder about its axis”.

In particular this means that:

- the centre of mass lies on the axis,
- two of the principal moments of inertia are the same and
- the principal axis corresponding to the other principal moment of inertia is the symmetry axis.

Making the mass matrix constant the same problem as balancing a rotor.
Dynamic Balancing

Can use some elementary ideas from Algebraic geometry to show that:

An arbitrary rigid-body can be balanced using two suitably chosen point-masses

To see this note that in $\mathbb{P}^9$ the set of $4 \times 4$ inertia matrices which are symmetric with respect to a given axis form a 3-plane, since they are determined by 6 linear equations.

Recall that the point masses form the Veronese variety in $\mathbb{P}^9$, the inertia matrices that can be formed by a pair of point masses lie on lines meeting the Veronese variety in a pair of points. The closure of this space of lines is called the secant variety to the Veronese variety.
Next given a particular inertia matrix \( \tilde{\Xi} \), the one to be balanced, we can take the set of 2-planes formed by the secant lines and \( \tilde{\Xi} \). The variety of all such planes is called the cone over the secant variety with vertex \( \tilde{\Xi} \).

The dimension of the cone over the secant variety is 7. Using naïve counting arguments the dimension of the secant variety would be 7 but this particular Veronese variety is well known to have a deficient secant variety, in fact the dimension is 6. Taking the cone over the secant variety adds another dimension.

The intersection of the cone over the secant variety with the 3-plane of symmetrical inertias will give points which specify how to balance \( \tilde{\Xi} \) with 2 point masses. From the above the intersection will have dimension 1 and hence there will be a one-parameter family of solutions.
Many other results in this area using more (or less) Algebraic geometry.
Conclusions

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THANK YOU